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Rate of Convergence of Some Neural Network Operators to the Unit-Univariate Case

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This article deals with the determination of the rate of convergence to the unit of some neural network operators, namely, the Cardaliaguet–Euvrard and "squashing" operators. This is given through the modulus of continuity of the involved function or its derivative and that appears in the right-hand side of the associated Jackson type inequality.

INTRODUCTION

The Cardaliaguet–Euvrard (1) operators were first introduced and studied extensively in [1], where the authors among many other things proved that these operators converge uniformly on compacta, to the unit over continuous and bounded functions. Our "squashing operator" (26) was motivated and inspired by the "squashing function" and related Theorem 6 of [1]. The work in [1] is qualitative where the used bell-shaped function is general. However, our work, though greatly motivated by [1], is quantitative and the used bell-shaped and "squashing" functions are of compact support. We produce a series of inequalities giving close upper bounds to the errors in approximating the unit operator by the above neutral network induced operators. All involved constants there are well determined. These are mainly pointwise estimates involving the first modulus of continuity of the engaged continuous function or some of its derivatives. For the above see Theorems 1, 2, 3, and 4. In Corollaries 1 and 2 we obtain some L_p-related estimates.

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I. CONVERGENCE WITH RATES OF THE CARDALIAGUET–EUVRARD NEURAL NETWORK OPERATORS

We need the following (see [1]).

DEFINITION 1. A function $b\colon \mathbf{R} \to \mathbf{R}$ is said to be *bell-shaped* if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to $\mathbf{R}$. In particular $b(x)$ is a nonnegative number and at a b takes a global maximum; it is the *center* of the bell-shaped function. A bell-shaped function is said to be *centered* if its center is zero. The function $b(x)$ may have jump discontinuities. In this work we consider only centered bell-shaped functions of compact support $[-T, T]$, $T > 0$. Call $I := \int_{-T}^{T} b(t)\,dt$. Note that $I > 0$.

EXAMPLES

(1) $b(x)$ can be the *characteristic* function over $[-1, 1]$.

(2) $b(x)$ can be the *hat* function over $[-1, 1]$, i.e.,

$$b(x) = \begin{cases} 1 + x, & -1 \le x \le 0, \\ 1 - x, & 0 < x \le 1, \\ 0, & \text{elsewhere}. \end{cases}$$

Here we consider functions $f\colon \mathbf{R} \to \mathbf{R}$ that are either continuous and bounded, or uniformly continuous.

In this article we study the pointwise convergence with rates over the real line, to the unit operator, of the *Cardaliaguet–Euvrard neural network operators* (see [1]),

$$(F_n(f))(x) := f_n(x) := \sum_{k=-n^2}^{n^2} \frac{f(k/n)}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right), \tag{1}$$

where $0 < \alpha < 1$ and $x \in \mathbf{R}$, $n \in \mathbf{N}$. The terms in the sum (1) can be nonzero iff

$$\left| n^{1-\alpha}\left(x - \frac{k}{n}\right)\right| \le T, \qquad \text{i.e.,} \left| x - \frac{k}{n} \right| \le \frac{T}{n^{1-\alpha}}$$

iff

$$nx - Tn^{\alpha} \le k \le nx + Tn^{\alpha}. \tag{2}$$

In order to have the desired order of numbers

$$-n^2 \le nx - Tn^{\alpha} \le nx + Tn^{\alpha} \le n^2, \tag{3}$$

it is sufficient enough to assume that

$$n \geq T + |x|. \tag{4}$$

When $x \in [-T, T]$ it is enough to assume $n \geq 2T$ which implies (3).

PROPOSITION 1. *Let* $a \leq b$, $a, b \in \mathbf{R}$. *Let* $\operatorname{card}(k)$ (≥ 0) *be the maximum number of integers contained in* $[a, b]$. *Then*

$$\max(0, (b-a) - 1) \leq \operatorname{card}(k) \leq (b-a) + 1.$$

Proof. Let the consecutive $k_1, \ldots, k_l \in \mathbf{Z}$ be in $[a, b]$, $l = \operatorname{card}(k) \geq 0$, such that $k_1 < k_2 < \cdots < k_l$. Here $k_{i+1} - k_i = 1$, $i = 1, \ldots, l-1$; $0 \leq k_1 - a \leq 1$, $0 \leq b - k_l \leq 1$. Thus

$$l - 1 \leq b - a = (k_1 - a) + (b - k_l) + \sum_{i=1}^{l-1} (k_{i+1} - k_i) \leq l + 1.$$

Hence

$$(b-a) - 1 \leq l.$$

And

$$l \leq (b-a) + 1. \quad \blacksquare$$

Note. We would like to establish a lower bound on $\operatorname{card}(k)$ over the interval $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$. From Proposition 1 we get that

$$\operatorname{card}(k) \geq \max(2Tn^{\alpha} - 1, 0).$$

We obtain $\operatorname{card}(k) \geq 1$, if

$$2Tn^{\alpha} - 1 \geq 1 \qquad \text{iff } n \geq T^{-1/\alpha}.$$

So to have the desired order (3) and $\operatorname{card}(k) \geq 1$ over $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$, we need to consider

$$n \geq \max(T + |x|, T^{-1/\alpha}). \tag{4*}$$

Also notice that $\operatorname{card}(k) \to +\infty$, as $n \to +\infty$. We call $b^* := b(0)$ the maximum of $b(x)$.

Denote by $[\cdot]$ the integral part of a number and by $\lceil \cdot \rceil$ its ceiling. Here comes our first main result.

THEOREM 1. *Let* $x \in \mathbf{R}$, $T > 0$, *and* $n \in \mathbf{N}$ *such that* $n \geq \max(T + |x|, T^{-1/\alpha})$. *Then*

$$|f_n(x) - f(x)| \leq |f(x)| \cdot \left| \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \frac{1}{I \cdot n^\alpha} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - 1 \right| + \frac{b^*}{I} \cdot \left(2T + \frac{1}{n^\alpha}\right) \cdot \omega_1\left(f, \frac{T}{n^{1-\alpha}}\right), \tag{5}$$

where ω_1 *is the first modulus of continuity of* f. *Inequality* (5) *becomes equality over constant functions*.

Proof. Note that

$$\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \frac{1}{I \cdot n^\alpha} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) \leq \frac{b^*}{I \cdot n^\alpha} \cdot \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} 1 \leq \frac{b^*}{I \cdot n^\alpha} \cdot (2Tn^\alpha + 1) = \frac{b^*}{I} \cdot \left(2T + \frac{1}{n^\alpha}\right). \tag{6}$$

Next we estimate

$$\begin{aligned}
\Delta &:= |f_n(x) - f(x)| = \left| \sum_{k=-n^2}^{n^2} \frac{f(k/n)}{I \cdot n^\alpha} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - f(x) \right| \\
&= \left| \sum_{k=-n^2}^{\lceil nx - Tn^\alpha \rceil - 1} \frac{f(k/n)}{I \cdot n^\alpha} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) \right. \\
&\quad + \sum_{k=[nx+Tn^\alpha]+1}^{n^2} \frac{f(k/n)}{I \cdot n^\alpha} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) \\
&\quad \left. + \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \frac{f(k/n)}{I \cdot n^\alpha} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - f(x) \right| \\
&= \left| \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \frac{f(k/n)}{I \cdot n^\alpha} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - f(x) \right|.
\end{aligned}$$

The last comes by the compact support $[-T, T]$ of b and (2); i.e., we get that

$$
\begin{aligned}
\Delta &= \left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{f(k/n)}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - f(x) \right| \\
&= \left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{f(k/n)}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) \right. \\
&\qquad - f(x) \cdot \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) \\
&\qquad \left. + f(x) \cdot \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - f(x) \right| \\
&\leq |f(x)| \cdot \left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - 1 \right| \\
&\qquad + \left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{(f(k/n) - f(x))}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) \right| \\
&\leq |f(x)| \cdot \left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - 1 \right| \\
&\qquad + \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{\omega_1(f, |k/n - x|)}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&|f_n(x) - f(x)| \\
&\quad \leq |f(x)| \cdot \left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - 1 \right| \\
&\qquad + \omega_1\left(f, \frac{T}{n^{1-\alpha}}\right) \cdot \left(\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) \right).
\end{aligned}
$$

Using now the upper bound (6) we obtain (5). ∎

We will use

LEMMA 1. *It holds that*

$$S_n(x) := \sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) \to 1,$$

pointwise, as $n \to +\infty$, *where* $x \in \mathbf{R}$.

Remark 1. Using Lemma 1 now, from inequality (5) we get that $f_n(x) \to f(x)$, pointwise with rates, as $n \to +\infty$, where $x \in \mathbf{R}$.

Proof of Lemma 1. Here $b(x)$ is nondecreasing over $[-T, 0]$ and nonincreasing over $[0, T]$.

(i) Case of $\lceil nx\rceil + 1 \le k \le [nx + Tn^{\alpha}]$; i.e., $nx < nx + 1 \le \lceil nx\rceil + 1 \le k \le [nx + Tn^{\alpha}]$. Let

$$nx \le k - 1 \le t \le k \Rightarrow x - \frac{k}{n} \le x - \frac{t}{n} \le x - \frac{(k-1)}{n} < 0.$$

Thus

$$b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) \le b\left(n^{1-\alpha}\left(x - \frac{t}{n}\right)\right) \le b\left(n^{1-\alpha}\left(x - \frac{(k-1)}{n}\right)\right).$$

Hence

$$b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) \le \int_{k-1}^{k} b\left(n^{1-\alpha}\left(x - \frac{t}{n}\right)\right) dt.$$

Now let $nx < k \le t \le k + 1$. Then

$$x - \frac{(k+1)}{n} \le x - \frac{t}{n} \le x - \frac{k}{n} < 0.$$

So that

$$b\left(n^{1-\alpha}\left(x - \frac{(k+1)}{n}\right)\right) \le b\left(n^{1-\alpha}\left(x - \frac{t}{n}\right)\right) \le b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right).$$

Therefore

$$\int_k^{k+1} b\left(n^{1-\alpha}\left(x - \frac{t}{n}\right)\right) \cdot dt \le b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right).$$

We have proved that

$$\int_k^{k+1} b\left(n^{1-\alpha}\left(x - \frac{t}{n}\right)\right) \cdot dt \le b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)$$
$$\le \int_{k-1}^{k} b\left(n^{1-\alpha}\left(x - \frac{t}{n}\right)\right) \cdot dt, \qquad (7)$$

for $\lceil nx \rceil + 1 \le k \le [nx + Tn^{\alpha}]$, where k is an integer.

(ii) Case of $\lceil nx - Tn^{\alpha} \rceil \le k \le [nx] - 1$; i.e., $k < k + 1 \le [nx] < nx$. From $k - 1 \le t \le k < nx$ we get

$$x - \frac{(k-1)}{n} \ge x - \frac{t}{n} \ge x - \frac{k}{n} > 0.$$

Thus

$$0 < b\left(n^{1-\alpha}\left(x - \frac{(k-1)}{n}\right)\right) \le b\left(n^{1-\alpha}\left(x - \frac{t}{n}\right)\right) \le b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right).$$

So that

$$\int_{k-1}^{k} b\left(n^{1-\alpha}\left(x - \frac{t}{n}\right)\right) \cdot dt \le b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right).$$

Let $k \le t \le k + 1 < nx$. Then

$$x - \frac{k}{n} \ge x - \frac{t}{n} \ge x - \frac{(k+1)}{n} > 0.$$

Hence

$$b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) \le b\left(n^{1-\alpha}\left(x - \frac{t}{n}\right)\right) \le b\left(n^{1-\alpha}\left(x - \frac{(k+1)}{n}\right)\right).$$

Therefore

$$b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right) \le \int_k^{k+1} b\left(n^{1-\alpha}\left(x - \frac{t}{n}\right)\right) \cdot dt.$$

We have established that

$$\int_{k-1}^{k} b\left(n^{1-\alpha}\left(x-\frac{t}{n}\right)\right)\cdot dt \le b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)$$
$$\le \int_{k}^{k+1} b\left(n^{1-\alpha}\left(x-\frac{t}{n}\right)\right)\cdot dt, \tag{8}$$

for $\lceil nx - Tn^{\alpha}\rceil \le k \le [nx]-1$, where k is an integer.

It is obvious that

$$\begin{aligned} S_n^3(x) &:= \frac{1}{I\cdot n^{\alpha}}\cdot b\left(n^{1-\alpha}\left(x-\frac{[nx]}{n}\right)\right) \to 0,\\ S_n^4(x) &:= \frac{1}{I\cdot n^{\alpha}}\cdot b\left(n^{1-\alpha}\left(x-\frac{\lceil nx\rceil}{n}\right)\right) \to 0, \end{aligned} \tag{9}$$

as $n \to +\infty$. Call

$$\begin{aligned} S_n^1(x) &:= \sum_{k=\lceil nx\rceil+1}^{[nx+Tn^{\alpha}]} \frac{1}{I\cdot n^{\alpha}}\cdot b\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right),\\ S_n^2(x) &:= \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx]-1} \frac{1}{I\cdot n^{\alpha}}\cdot b\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right). \end{aligned} \tag{10}$$

So that

$$S_n(x) := S_n^1(x) + S_n^2(x) + S_n^3(x) + S_n^4(x). \tag{11}$$

From (7) we obtain that

$$\frac{1}{I\cdot n^{\alpha}}\cdot\int_{\lceil nx\rceil+1}^{[nx+Tn^{\alpha}]+1} b\left(n^{1-\alpha}\left(x-\frac{t}{n}\right)\right)\cdot dt$$
$$\le S_n^1(x) \le \frac{1}{I\cdot n^{\alpha}}\cdot\int_{\lceil nx\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{t}{n}\right)\right) dt;$$

i.e.,

$$\frac{1}{I}\cdot\int_{n^{1-\alpha}(x-([nx+Tn^{\alpha}]+1)/n)}^{n^{1-\alpha}\cdot(x-(\lceil nx\rceil+1)/n)} b(t)\,dt$$
$$\le S_n^1(x) \le \frac{1}{I}\cdot\int_{n^{1-\alpha}\cdot(x-[nx+Tn^{\alpha}]/n)}^{n^{1-\alpha}\cdot(x-\lceil nx\rceil/n)} b(t)\,dt. \tag{12}$$

But we observe that

$$\lim_{n\to+\infty}\left(\frac{nx-\lceil nx\rceil-1}{n^{\alpha}}\right)=\lim_{n\to+\infty}\left(\frac{nx-\lceil nx\rceil}{n^{\alpha}}\right)=0,$$

also that

$$\lim_{n\to+\infty}\left(\frac{nx-[nx+Tn^{\alpha}]-1}{n^{\alpha}}\right)=\lim_{n\to+\infty}\left(\frac{nx-[nx+Tn^{\alpha}]}{n^{\alpha}}\right)=-T.$$

Therefore we get

$$S_n^1(x)\to\frac{1}{I}\cdot\int_{-T}^{0}b(t)\,dt,\qquad\text{as } n\to+\infty.$$

Likewise from (8) we get that

$$\begin{aligned}\frac{1}{I\cdot n^{\alpha}}&\cdot\int_{\lceil nx-Tn^{\alpha}\rceil-1}^{[nx]-1}b\left(n^{1-\alpha}\left(x-\frac{t}{n}\right)\right)\cdot dt\\&\le S_n^2(x)\le\frac{1}{I\cdot n^{\alpha}}\cdot\int_{\lceil nx-Tn^{\alpha}\rceil}^{[nx]}b\left(n^{1-\alpha}\left(x-\frac{t}{n}\right)\right)\cdot dt;\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{1}{I}\cdot\int_{n^{1-\alpha}\cdot(x-([nx]-1)/n)}^{n^{1-\alpha}\cdot(x-(\lceil nx-Tn^{\alpha}\rceil-1)/n)}b(t)\,dt\le S_n^2(x)&\\\le\frac{1}{I}\cdot\int_{n^{1-\alpha}\cdot(x-[nx]/n)}^{n^{1-\alpha}\cdot(x-\lceil nx-Tn^{\alpha}\rceil/n)}b(t)\,dt.&\end{aligned}$$

Since

$$\lim_{n\to+\infty}\left(\frac{nx-\lceil nx-Tn^{\alpha}\rceil+1}{n^{\alpha}}\right)=\lim_{n\to+\infty}\left(\frac{nx-\lceil nx-Tn^{\alpha}\rceil}{n^{\alpha}}\right)=T,$$

and

$$\lim_{n\to+\infty}\left(\frac{nx-[nx]+1}{n^{\alpha}}\right)=\lim_{n\to+\infty}\left(\frac{nx-[nx]}{n^{\alpha}}\right)=0,$$

we obtain

$$S_n^2(x)\to\frac{1}{I}\cdot\int_{0}^{T}b(t)\,dt,\qquad\text{as } n\to+\infty.$$

We have established now that

$$\lim_{n \to +\infty} S_n(x) = 1, \qquad x \in \mathbf{R}. \qquad \blacksquare$$

Our second main result follows.

THEOREM 2. *Let $x \in \mathbf{R}$, $T > 0$, and $n \in \mathbf{N}$ such that $n \geq \max(T + |x|, T^{-1/\alpha})$. Let $f \in C^N(\mathbf{R})$, $N \in \mathbf{N}$, such that $f^{(N)}$ is a uniformly continuous function or $f^{(N)}$ is continuous and bounded. Then*

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \sum_{k=-n^2}^{n^2} \frac{f(k/n)}{I \cdot n^\alpha} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - f(x) \right| \\ &\leq |f(x)| \cdot \left| \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} \frac{1}{I \cdot n^\alpha} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - 1 \right| \\ &\quad + \frac{b^*}{I} \cdot \left(2T + \frac{1}{n^\alpha}\right) \cdot \left(\sum_{j=1}^{N} \frac{|f^{(j)}(x)| \cdot T^j}{n^{j \cdot (1-\alpha)} \cdot j!} \right) \\ &\quad + \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! \cdot n^{N \cdot (1-\alpha)}} \cdot \frac{b^*}{I} \cdot \left(2T + \frac{1}{n^\alpha}\right). \end{aligned} \tag{13}$$

Remark 2. Inequality (13) is attained by constant functions. Also notice that as $n \to +\infty$ we have that R.H.S. (13) $\to 0$ (see Lemma 1, etc.), therefore L.H.S. (13) $\to 0$; i.e., (13) gives us with rates the pointwise convergence of $f_n(x) \to f(x)$, as $n \to +\infty$, $x \in \mathbf{R}$.

Proof of Theorem 2. Note that b here is of compact support $[-T, T]$ and all assumptions are as earlier. By Taylor's formula we have that

$$\begin{aligned} f\left(\frac{k}{n}\right) &= \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \cdot \left(\frac{k}{n} - x\right)^j \\ &\quad + \int_x^{k/n} \left(f^{(N)}(t) - f^{(N)}(x)\right) \cdot \frac{((k/n) - t)^{N-1}}{(N-1)!} \cdot dt. \end{aligned}$$

Hence

$$\frac{f(k/n)\cdot b\left(n^{1-\alpha}(x-k/n)\right)}{I\cdot n^{\alpha}}$$
$$=\sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}\cdot\left(\frac{k}{n}-x\right)^{j}$$
$$\cdot\frac{b\left(n^{1-\alpha}\cdot(x-k/n)\right)}{I\cdot n^{\alpha}}+\frac{b\left(n^{1-\alpha}\cdot(x-k/n)\right)}{I\cdot n^{\alpha}}$$
$$\cdot\int_{x}^{k/n}\left(f^{(N)}(t)-f^{(N)}(x)\right)\cdot\frac{(k/n-t)^{N-1}}{(N-1)!}\cdot dt.$$

Thus

$$(F_n(f))(x)-f(x)$$
$$=\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{f(k/n)\cdot b\left(n^{1-\alpha}\cdot(x-k/n)\right)}{I\cdot n^{\alpha}}-f(x)$$
$$=-f(x)+\sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}$$
$$\cdot\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{(k/n-x)^{j}}{I\cdot n^{\alpha}}\cdot b\left(n^{1-\alpha}\cdot(x-k/n)\right)\right)+R=:\otimes,$$

where

$$R:=\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\cdot(x-k/n)\right)}{I\cdot n^{\alpha}}$$
$$\cdot\int_{x}^{k/n}\left(f^{(N)}(t)-f^{(N)}(x)\right)\cdot\frac{(k/n-t)^{N-1}}{(N-1)!}\cdot dt, \tag{14}$$

$$\otimes=-f(x)+f(x)\cdot\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{1}{I\cdot n^{\alpha}}\cdot b\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right)\right)$$
$$+\sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}$$
$$\cdot\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{(k/n-x)^{j}}{I\cdot n^{\alpha}}\cdot b\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right)\right)+R.$$

Therefore

$$
\begin{aligned}
&(F_n(f))(x) - f(x) \\
&\quad = f(x) \cdot \left(\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{1}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) - 1 \right) \\
&\qquad + \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \\
&\qquad \cdot \left(\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{(k/n - x)^j}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) \right) + R.
\end{aligned}
$$

So that

$$
\begin{aligned}
&|(F_n(f))(x) - f(x)| \\
&\quad \leq |f(x)| \cdot \left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{1}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) - 1 \right| \\
&\qquad + \sum_{j=1}^{N} \frac{|f^{(j)}(x)|}{j!} \\
&\qquad \cdot \left(\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{T^j}{n^{j \cdot (1-\alpha)} \cdot I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) \right) + |R|.
\end{aligned}
$$

And

$$
\begin{aligned}
&|(F_n(f))(x) - f(x)| \qquad (\text{by } (6)) \\
&\quad \leq |f(x)| \cdot \left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{1}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) - 1 \right| \\
&\qquad + \sum_{j=1}^{N} \frac{|f^{(j)}(x)|}{j!} \cdot \frac{T^j}{n^{j \cdot (1-\alpha)}} \cdot \frac{b^*}{I} \cdot \left(2T + \frac{1}{n^{\alpha}} \right) + |R| \\
&\quad = |f(x)| \cdot \left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{1}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) - 1 \right| \\
&\qquad + \frac{b^*}{I} \cdot \left(2T + \frac{1}{n^{\alpha}} \right) \cdot \left(\sum_{j=1}^{N} \frac{T^j \cdot |f^{(j)}(x)|}{n^{j \cdot (1-\alpha)} \cdot j!} \right) + |R|. \qquad (15)
\end{aligned}
$$

Next we estimate

$$|R| = \left| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\cdot(x-k/n)\right)}{I\cdot n^{\alpha}} \cdot \int_x^{k/n} \left(f^{(N)}(t) - f^{(N)}(x)\right)\cdot \frac{(k/n-t)^{N-1}}{(N-1)!}\cdot dt \right|$$

$$\le \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\cdot(x-k/n)\right)}{I\cdot n^{\alpha}} \cdot \left| \int_x^{k/n} \left(f^{(N)}(t) - f^{(N)}(x)\right)\cdot \frac{(k/n-t)^{N-1}}{(N-1)!}\cdot dt \right|$$

$$\le \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\cdot(x-k/n)\right)}{I\cdot n^{\alpha}} \cdot \gamma \le (*),$$

where

$$\gamma := \left| \int_x^{k/n} |f^{(N)}(t) - f^{(N)}(x)| \cdot \frac{|k/n-t|^{N-1}}{(N-1)!}\cdot dt \right| \tag{16}$$

$$(*) \le \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\cdot(x-k/n)\right)}{I\cdot n^{\alpha}} \cdot \varphi, \tag{17}$$

where

$$\varphi := \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N!\cdot n^{N\cdot(1-\alpha)}}. \tag{18}$$

The last part of inequality (17) comes from the following:

(i) Let $x \le k/n$, then

$$\gamma = \int_x^{k/n} |f^{(N)}(t) - f^{(N)}(x)| \cdot \frac{|k/n-t|^{N-1}}{(N-1)!}\cdot dt$$

$$\le \int_x^{k/n} \omega_1\left(f^{(N)}, |t-x|\right) \cdot \frac{|k/n-t|^{N-1}}{(N-1)!}\cdot dt$$

$$\le \omega_1\left(f^{(N)}, \left|x - \frac{k}{n}\right|\right) \cdot \int_x^{k/n} \frac{(k/n-t)^{N-1}}{(N-1)!}\cdot dt$$

$$\le \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{(k/n-x)^N}{N!} \le \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N!\cdot n^{N\cdot(1-\alpha)}};$$

i.e., when $x \le k/n$ we get

$$\gamma \le \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! \cdot n^{N\cdot(1-\alpha)}}. \tag{19}$$

(ii) Let $x \ge k/n$, then

$$\begin{aligned}
\gamma &= \left| \int_{k/n}^{x} |f^{(N)}(t) - f^{(N)}(x)| \cdot \frac{|t-k/n|^{N-1}}{(N-1)!} \cdot dt \right| \\
&= \int_{k/n}^{x} |f^{(N)}(t) - f^{(N)}(x)| \cdot \frac{(t-k/n)^{N-1}}{(N-1)!} \cdot dt \\
&\le \int_{k/n}^{x} \omega_1(f^{(N)}, |t-x|) \cdot \frac{(t-k/n)^{N-1}}{(N-1)!} \cdot dt \\
&\le \omega_1\left(f^{(N)}, \left|x - \frac{k}{n}\right|\right) \cdot \int_{k/n}^{x} \frac{(t-k/n)^{N-1}}{(N-1)!} \cdot dt \\
&= \omega_1\left(f^{(N)}, \left|x - \frac{k}{n}\right|\right) \cdot \frac{(x-k/n)^{N}}{N!} \le \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! \cdot n^{N\cdot(1-\alpha)}}.
\end{aligned}$$

Thus in both cases we have

$$\gamma \le \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! \cdot n^{N\cdot(1-\alpha)}}. \tag{20}$$

Consequently from (6), (17), and (20) we obtain

$$|R| \le \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! \cdot n^{N\cdot(1-\alpha)}} \cdot \frac{b^*}{I} \cdot \left(2T + \frac{1}{n^{\alpha}}\right). \tag{21}$$

Finally from (15) and (21) we conclude inequality (13). ∎

COROLLARY 1. *Let $b(x)$ be a centered bell-shaped continuous function on* $\mathbf{R}$ *of compact support* $[-T, T]$. *Let* $x \in [-T^*, T^*]$, $T^* > 0$, *and* $n \in \mathbf{N}$ *be such that* $n \ge \max(T + T^*, T^{-1/\alpha})$, $0 < \alpha < 1$. *Consider* $p \ge 1$. *Then*

$$\begin{aligned}
&\|f_n - f\|_{p,[-T^*,T^*]} \\
&\quad \le \|f\|_{\infty,[-T^*,T^*]} \\
&\qquad \cdot \left\| \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I \cdot n^{\alpha}} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - 1 \right\|_{p,[-T^*,T^*]} \\
&\qquad + \omega_1\left(f, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{b^*}{I} \cdot \left(2T + \frac{1}{n^{\alpha}}\right) \cdot 2^{1/p} \cdot T^{*1/p},
\end{aligned} \tag{22}$$

where $I := \int_{-T}^{T} b(t)\, dt$.

Inequality (22) is attained by constant functions. Furthermore from (22) we get the L_p convergence of f_n to f with rates.

Proof. From Theorem 1(5) we have

$$|f_n(x) - f(x)| \leq \|f\|_{\infty,[-T^*,T^*]} \cdot \left| \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \frac{1}{I \cdot n^\alpha} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - 1 \right| + \frac{b^*}{I} \cdot \left(2T + \frac{1}{n^\alpha}\right) \cdot \omega_1\left(f, \frac{T}{n^{1-\alpha}}\right). \tag{23}$$

Inequality (22) now comes by integration of (23) and the properties of the L_p-norm. From (6) we obtain that

$$\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \frac{1}{I \cdot n^\alpha} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) \leq \frac{b^*}{I} \cdot (2T + 1).$$

Thus

$$\left| \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \frac{1}{I \cdot n^\alpha} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - 1 \right|^p \leq \left(\frac{b^*}{I} \cdot (2T + 1) + 1\right)^p =: M > 0,$$

for all $n \in \mathbf{N}$, and all $x \in [-T^*, T^*]$. Also, from Lemma 1, we get that

$$\lim_{n \to +\infty} \left| \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \frac{1}{I \cdot n^\alpha} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - 1 \right|^p = 0,$$

all $x \in [-T^*, T^*]$. Now it is clear from the bounded convergence theorem that

$$\lim_{n \to \infty} \left\| \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \frac{1}{I \cdot n^\alpha} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - 1 \right\|_{p,[-T^*,T^*]} = 0. \qquad \blacksquare$$

COROLLARY 2. *Let $b(x)$ be a centered bell-shaped continuous function on* $\mathbf{R}$ *of compact support* $[-T, T]$. *Let* $x \in [-T^*, T^*]$, $T^* > 0$, *and* $n \in \mathbf{N}$ *be such that* $n \geq \max(T + T^*, T^{-1/\alpha})$, $0 < \alpha < 1$. *Consider* $p \geq 1$. *Then*

$$\begin{aligned}
&\|f_n - f\|_{p,[-T^*,T^*]} \\
&\qquad \leq \|f\|_{\infty,[-T^*,T^*]} \\
&\qquad\quad \cdot \left\| \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} \frac{1}{I \cdot n^\alpha} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - 1 \right\|_{p,[-T^*,T^*]} \\
&\qquad\quad + \frac{b^*}{I} \cdot \left(2T + \frac{1}{n^\alpha}\right) \cdot \left(\sum_{j=1}^{N} \frac{T^j \cdot \|f^{(j)}\|_{p,[-T^*,T^*]}}{n^{j\cdot(1-\alpha)} \cdot j!} \right) \\
&\qquad\quad + \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{2^{1/p} \cdot T^N \cdot T^{*1/p}}{N! \cdot n^{N\cdot(1-\alpha)}} \cdot \frac{b^*}{I} \cdot \left(2T + \frac{1}{n^\alpha}\right), \qquad (24)
\end{aligned}$$

where $N \geq 1$.

The last is attained by constant functions. Here from (24) we get again the L_p convergence of f_n to f with rates.

Proof. From Theorem 2(13) we have that

$$\begin{aligned}
&|f_n(x) - f(x)| \\
&\qquad \leq \|f\|_{\infty,[-T^*,T^*]} \cdot \left| \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} \frac{1}{I \cdot n^\alpha} \cdot b\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - 1 \right| \\
&\qquad\quad + \frac{b^*}{I} \cdot \left(2T + \frac{1}{n^\alpha}\right) \cdot \left(\sum_{j=1}^{n} \frac{|f^{(j)}(x)| \cdot T^j}{n^{j\cdot(1-\alpha)} j!} \right) \\
&\qquad\quad + \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{T^N}{N! \cdot n^{N\cdot(1-\alpha)}} \cdot \frac{b^*}{I} \cdot \left(2T + \frac{1}{n^\alpha}\right). \qquad (25)
\end{aligned}$$

Inequality (24) now comes by integration of (25) and the properties of the L_p-norm. ∎

II. THE "SQUASHING OPERATORS" AND THEIR CONVERGENCE TO THE UNIT WITH RATES

We need

DEFINITION 2. Let the nonnegative function S: $\mathbf{R} \to \mathbf{R}$, S has compact support $[-T, T]$, $T > 0$, and is nondecreasing there and it can be continuous only on either $(-\infty, T]$ or $[-T, T]$. S can have jump discontinuities. We call S the "squashing function" (see also [1]).

Let $f\colon \mathbf{R} \to \mathbf{R}$ be either uniformly continuous or continuous and bounded. Assume that

$$I^* := \int_{-T}^{T} S(t)\, dt > 0.$$

Obviously

$$\max_{x \in [-T, T]} S(x) = S(T).$$

For $x \in \mathbf{R}$ we define the "squashing operator"

$$(G_n(f))(x) := \sum_{k=-n^2}^{n^2} \frac{f(k/n)}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right), \tag{26}$$

$0 < \alpha < 1$ and $n \in \mathbf{N}$: $n \geq \max(T + |x|, T^{-1/\alpha})$. It is clear that

$$(G_n(f))(x) = \sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx + Tn^{\alpha}]} \frac{f(k/n)}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right). \tag{27}$$

Here we study the pointwise convergence with rates of $(G_n(f))(x) \to f(x)$, as $n \to +\infty$, $x \in \mathbf{R}$.

THEOREM 3. *Under the above terms and assumptions we obtain*

$$\begin{aligned} &|(G_n(f))(x) - f(x)| \\ &\quad \leq |f(x)| \cdot \left| \sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx + Tn^{\alpha}]} \frac{1}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) - 1 \right| \\ &\qquad + \omega_1\left(f, \frac{T}{n^{1-\alpha}}\right) \cdot \frac{S(T)}{I^*} \cdot \left(2T + \frac{1}{n^{\alpha}}\right). \end{aligned} \tag{28}$$

Inequality (28) is attained by constant functions.

Proof. Notice that

$$
\begin{aligned}
&|(G_n f)(x) - f(x)| \\
&\quad = \left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{f(k/n)}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) - f(x) \right| \\
&\quad = \left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{(f(k/n) - f(x))}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) \right. \\
&\qquad \left. + f(x) \cdot \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{1}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) - f(x) \right| \\
&\quad \leq \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{|f(k/n) - f(x)|}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) \\
&\qquad + |f(x)| \cdot \left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{1}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) - 1 \right| \\
&\quad \leq \omega_1\left(f, \frac{T}{n^{1-\alpha}} \right) \cdot \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{1}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) \\
&\qquad + |f(x)| \cdot \left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{1}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) - 1 \right|.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&|(G_n f)(x) - f(x)| \\
&\quad \leq |f(x)| \cdot \left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{1}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) - 1 \right| \\
&\qquad + \omega_1\left(f, \frac{T}{n^{1-\alpha}} \right) \cdot \frac{S(T)}{I^* \cdot n^{\alpha}} \cdot \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} 1.
\end{aligned}
$$

Using in the last that

$$
\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} 1 \leq (2Tn^{\alpha} + 1)
$$

we obtain (28). ∎

LEMMA 2. *It holds that*

$$D_n(x) := \sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{1}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) \to 1,$$

pointwise, as $n \to +\infty$, *where* $x \in \mathbf{R}$.

Remark 3. Using Lemma 2 now, from inequality (28) we get $(G_n(f))(x) \to f(x)$, pointwise with rates, as $n \to +\infty$, where $x \in \mathbf{R}$.

Proof of Lemma 2. (i) Case of $\lceil nx \rceil + 1 \le k \le [nx + Tn^{\alpha}]$, i.e.,

$$nx < nx + 1 \le \lceil nx \rceil + 1 \le k \le [nx + Tn^{\alpha}].$$

Consider t,

$$nx \le k - 1 \le t \le k,$$

that is,

$$x - \frac{k}{n} \le x - \frac{t}{n} \le x - \frac{(k-1)}{n} < 0.$$

Since S is nondecreasing we get

$$0 \le S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) \le S\left(n^{1-\alpha} \cdot \left(x - \frac{t}{n}\right)\right)$$
$$\le S\left(n^{1-\alpha} \cdot \left(x - \frac{(k-1)}{n}\right)\right).$$

Hence

$$S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right) \le \int_{k-1}^{k} S\left(n^{1-\alpha} \cdot \left(x - \frac{t}{n}\right)\right) \cdot dt.$$

Let now $nx < k \le t \le k + 1$, then

$$x - \frac{(k+1)}{n} \le x - \frac{t}{n} \le x - \frac{k}{n} < 0.$$

Therefore

$$0 \le S\left(n^{1-\alpha} \cdot \left(x - \frac{(k+1)}{n}\right)\right) \le S\left(n^{1-\alpha} \cdot \left(x - \frac{t}{n}\right)\right)$$
$$\le S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right).$$

Consequently we obtain

$$\int_k^{k+1} S\left(n^{1-\alpha}\cdot\left(x-\frac{t}{n}\right)\right)dt \le S\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right);$$

i.e., we have gotten

$$\int_k^{k+1} S\left(n^{1-\alpha}\cdot\left(x-\frac{t}{n}\right)\right)dt \le S\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right) \le \int_{k-1}^{k} S\left(n^{1-\alpha}\cdot\left(x-\frac{t}{n}\right)\right)\cdot dt, \quad (29)$$

for any integer k, $\lceil nx\rceil + 1 \le k \le [nx + Tn^{\alpha}]$.

(ii) Case of $\lceil nx - Tn^{\alpha}\rceil \le k \le [nx] - 1$. Here

$$k < k+1 \le [nx] < nx.$$

Consider t,

$$k-1 \le t \le k < nx,$$

then

$$x - \frac{(k-1)}{n} \ge x - \frac{t}{n} \ge x - \frac{k}{n} > 0.$$

Since S is nondecreasing we get

$$S\left(n^{1-\alpha}\cdot\left(x-\frac{(k-1)}{n}\right)\right) \ge S\left(n^{1-\alpha}\cdot\left(x-\frac{t}{n}\right)\right) \ge S\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right) \ge 0.$$

Thus

$$\int_{k-1}^{k} S\left(n^{1-\alpha}\cdot\left(x-\frac{t}{n}\right)\right)\cdot dt \ge S\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right).$$

Let $k \le t \le k+1 < nx$, then

$$x - \frac{k}{n} \ge x - \frac{t}{n} \ge x - \frac{(k+1)}{n} > 0.$$

Therefore

$$S\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right)\geq S\left(n^{1-\alpha}\cdot\left(x-\frac{t}{n}\right)\right)$$
$$\geq S\left(n^{1-\alpha}\cdot\left(x-\frac{(k+1)}{n}\right)\right)\geq 0.$$

That is,

$$S\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right)\geq\int_k^{k+1}S\left(n^{1-\alpha}\cdot\left(x-\frac{t}{n}\right)\right)\cdot dt.$$

We have proved that

$$\int_k^{k+1}S\left(n^{1-\alpha}\cdot\left(x-\frac{t}{n}\right)\right)\cdot dt\leq S\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right)$$
$$\leq\int_{k-1}^{k}S\left(n^{1-\alpha}\cdot\left(x-\frac{t}{n}\right)\right)\cdot dt,\quad(30)$$

for any integer k,

$$\lceil nx-Tn^{\alpha}\rceil\leq k\leq[nx]-1.$$

Notice that for specific k^* we get

$$0\leq\frac{1}{I^*\cdot n^{\alpha}}\cdot S\left(n^{1-\alpha}\cdot\left(x-\frac{k^*}{n}\right)\right)\leq\frac{S(T)}{I^*\cdot n^{\alpha}}\to 0,$$

as $n\to+\infty$.

Hence

$$D_n^3(x):=\frac{1}{I^*\cdot n^{\alpha}}\cdot S\left(n^{1-\alpha}\cdot\left(x-\frac{[nx]}{n}\right)\right)\to 0$$

and

$$D_n^4(x):=\frac{1}{I^*\cdot n^{\alpha}}\cdot S\left(n^{1-\alpha}\cdot\left(x-\frac{\lceil nx\rceil}{n}\right)\right)\to 0,$$

as $n\to+\infty$. Denote by

$$D_n^1(x):=\sum_{k=\lceil nx\rceil+1}^{[nx+Tn^{\alpha}]}\frac{1}{I^*\cdot n^{\alpha}}\cdot S\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right)$$

and

$$D_n^2(x) := \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx]-1} \frac{1}{I^* \cdot n^\alpha} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right).$$

So we can write

$$D_n(x) = D_n^1(x) + D_n^2(x) + D_n^3(x) + D_n^4(x).$$

From (29) we find

$$\frac{1}{I^* \cdot n^\alpha} \cdot \int_{\lceil nx \rceil + 1}^{[nx + Tn^\alpha] + 1} S\left(n^{1-\alpha} \cdot \left(x - \frac{t}{n}\right)\right) \cdot dt$$
$$\leq D_n^1(x) \leq \frac{1}{I^* \cdot n^\alpha} \cdot \int_{\lceil nx \rceil}^{[nx + Tn^\alpha]} S\left(n^{1-\alpha} \cdot \left(x - \frac{t}{n}\right)\right) \cdot dt.$$

Hence

$$\frac{1}{I^*} \cdot \int_{n^{1-\alpha} \cdot (x - ([nx + Tn^\alpha] + 1)/n)}^{n^{1-\alpha} \cdot (x - (\lceil nx \rceil + 1)/n)} S(t) \cdot dt$$
$$\leq D_n^1(x) \leq \frac{1}{I^*} \cdot \int_{n^{1-\alpha} \cdot (x - [nx + Tn^\alpha]/n)}^{n^{1-\alpha} \cdot (x - \lceil nx \rceil / n)} S(t)\, dt.$$

Since

$$\lim_{n \to +\infty} \left(\frac{nx - \lceil nx \rceil - 1}{n^\alpha} \right) = \lim_{n \to +\infty} \left(\frac{nx - \lceil nx \rceil}{n^\alpha} \right) = 0$$

and

$$\lim_{n \to +\infty} \left(\frac{nx - [nx + Tn^\alpha] - 1}{n^\alpha} \right) = \lim_{n \to +\infty} \left(\frac{nx - [nx + Tn^\alpha]}{n^\alpha} \right) = -T,$$

we get that

$$\lim_{n \to +\infty} D_n^1(x) = \frac{1}{I^*} \cdot \int_{-T}^{0} S(t) \cdot dt. \tag{31}$$

Likewise we obtain from (30) that

$$\frac{1}{I^* \cdot n^\alpha} \cdot \int_{\lceil nx - Tn^\alpha \rceil}^{[nx]} S\left(n^{1-\alpha} \cdot \left(x - \frac{t}{n}\right)\right) \cdot dt$$
$$\leq D_n^2(x) \leq \frac{1}{I^* \cdot n^\alpha} \cdot \int_{\lceil nx - Tn^\alpha \rceil - 1}^{[nx]-1} S\left(n^{1-\alpha} \cdot \left(x - \frac{t}{n}\right)\right) \cdot dt.$$

That is,

$$\frac{1}{I^*}\cdot\int_{n^{1-\alpha}\cdot(x-[nx]/n)}^{n^{1-\alpha}\cdot(x-\lceil nx-Tn^{\alpha}\rceil/n)} S(t)\cdot dt \le D_n^2(x)$$
$$\le \frac{1}{I^*}\cdot\int_{n^{1-\alpha}\cdot(x-([nx]-1)/n)}^{n^{1-\alpha}\cdot(x-(\lceil nx-Tn^{\alpha}\rceil-1)/n)} S(t)\cdot dt.$$

Since

$$\lim_{n\to+\infty}\left(\frac{nx-[nx]}{n^{\alpha}}\right)=\lim_{n\to+\infty}\left(\frac{nx-[nx]+1}{n^{\alpha}}\right)=0,$$

and

$$\lim_{n\to+\infty}\left(\frac{nx-\lceil nx-Tn^{\alpha}\rceil}{n^{\alpha}}\right)=\lim_{n\to+\infty}\left(\frac{nx-\lceil nx-Tn^{\alpha}\rceil+1}{n^{\alpha}}\right)=T,$$

we find that

$$\lim_{n\to+\infty} D_n^2(x)=\frac{1}{I^*}\cdot\int_0^T S(t)\cdot dt. \tag{32}$$

Finally, from all the above, the representation of $D_n(x)$, and (31), (32), we conclude that $\lim_{n\to+\infty} D_n(x)=1$, pointwise, $x\in\mathbf{R}$. ∎

As a related result we give

THEOREM 4. *Let $x\in\mathbf{R}$, $T>0$, and $n\in\mathbf{N}$, such that $n\ge\max(T+|x|, T^{-1/\alpha})$. Let $f\in C^N(\mathbf{R})$, $N\in\mathbf{N}$, such that $f^{(N)}$ is a uniformly continuous function or $f^{(N)}$ is continuous and bounded. Then*

$$|(G_n(f))(x)-f(x)|$$
$$=\left|\sum_{k=-n^2}^{n^2}\frac{f(k/n)}{I^*\cdot n^{\alpha}}\cdot S\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right)-f(x)\right|$$
$$\le |f(x)|\cdot\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{1}{I^*\cdot n^{\alpha}}\cdot S\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right)-1\right|$$
$$+\frac{S(T)}{I^*}\cdot\left(2T+\frac{1}{n^{\alpha}}\right)\cdot\left(\sum_{j=1}^{N}\frac{|f^{(j)}(x)|\cdot T^j}{j!\cdot n^{j\cdot(1-\alpha)}}\right)$$
$$+\omega_1\left(f^{(N)},\frac{T}{n^{1-\alpha}}\right)\cdot\frac{T^N}{N!\cdot n^{N\cdot(1-\alpha)}}\cdot\frac{S(T)}{I^*}\cdot\left(2T+\frac{1}{n^{\alpha}}\right). \tag{33}$$

Inequality (33) is attained by constant functions. Also, (33) gives us with rates (see Lemma 2), the pointwise convergence of $G_n(f)(x) \to f(x)$, as $n \to +\infty$, $x \in \mathbf{R}$.

Proof. As in the proof of Theorem 2 we have

$$\frac{f(k/n)}{I^* \cdot n^\alpha} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)$$
$$= \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \cdot \left(\frac{k}{n} - x\right)^j \cdot \frac{S\left(n^{1-\alpha} \cdot (x - k/n)\right)}{I^* \cdot n^\alpha}$$
$$+ \frac{S\left(n^{1-\alpha} \cdot (x - k/n)\right)}{I^* \cdot n^\alpha}$$
$$\cdot \int_x^{k/n} \left(f^{(N)}(t) - f^{(N)}(x)\right) \cdot \frac{(k/n - t)^{N-1}}{(N-1)!} \cdot dt.$$

Thus

$$(G_n(f))(x) - f(x)$$
$$= \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} f\left(\frac{k}{n}\right) \cdot \frac{S\left(n^{1-\alpha} \cdot (x - k/n)\right)}{I^* \cdot n^\alpha} - f(x)$$
$$= -f(x) + \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!}$$
$$\cdot \left(\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} \frac{(k/n - x)^j}{I^* \cdot n^\alpha} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n}\right)\right)\right) + R^* =: \otimes.$$

Here

$$R^* := \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} \frac{S\left(n^{1-\alpha} \cdot (x - k/n)\right)}{I^* \cdot n^\alpha}$$
$$\cdot \int_x^{k/n} \left(f^{(N)}(t) - f^{(N)}(x)\right) \cdot \frac{(k/n - t)^{N-1}}{(N-1)!} \cdot dt;$$

i.e.,

$$\otimes = f(x) \cdot \left(\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{1}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) - 1 \right) + \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!}$$

$$\cdot \left(\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{(k/n - x)^j}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) \right) + R^*.$$

That is,

$$\begin{aligned}
&|(G_n(f))(x) - f(x)| \\
&\quad \leq |f(x)| \cdot \left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{1}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) - 1 \right| \\
&\qquad + \sum_{j=1}^{N} \frac{|f^{(j)}(x)|}{j!} \\
&\qquad \cdot \left(\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{(T^j / n^{j \cdot (1-\alpha)})}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) \right) + |R^*| \\
&\quad \leq \quad (\text{by (6) for } S) \\
&\qquad |f(x)| \cdot \left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{1}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) - 1 \right| \\
&\qquad + \sum_{j=1}^{N} \frac{|f^{(j)}(x)|}{j!} \cdot \frac{T^j}{n^{j \cdot (1-\alpha)}} \cdot \frac{S(T)}{I^*} \cdot \left(2T + \frac{1}{n^{\alpha}} \right) + |R^*|.
\end{aligned}$$

Hence

$$\begin{aligned}
&|(G_n(f))(x) - f(x)| \\
&\quad \leq |f(x)| \cdot \left| \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} \frac{1}{I^* \cdot n^{\alpha}} \cdot S\left(n^{1-\alpha} \cdot \left(x - \frac{k}{n} \right) \right) - 1 \right| \\
&\qquad + \frac{S(T)}{I^*} \cdot \left(2T + \frac{1}{n^{\alpha}} \right) \cdot \left(\sum_{j=1}^{N} \frac{|f^{(j)}(x)| \cdot T^j}{j! \cdot n^{j \cdot (1-\alpha)}} \right) + |R^*|. \qquad (34)
\end{aligned}$$

We would like to estimate

$$|R^*| \leq \sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \frac{S\left(n^{1-\alpha}\cdot(x-k/n)\right)}{I^*\cdot n^\alpha} \cdot\left|\int_x^{k/n}\left(f^{(N)}(t)-f^{(N)}(x)\right)\cdot\frac{(k/n-t)^{N-1}}{(N-1)!}\cdot dt\right|$$

$$\leq \sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \frac{S\left(n^{1-\alpha}\cdot(x-k/n)\right)}{I^*\cdot n^\alpha}\cdot\gamma =: \otimes.$$

Here

$$\gamma := \left|\int_x^{k/n}|f^{(N)}(t)-f^{(N)}(x)|\cdot\frac{|k/n-t|^{N-1}}{(N-1)!}\cdot dt\right|$$

$$\leq \omega_1\left(f^{(N)},\frac{T}{n^{1-\alpha}}\right)\cdot\frac{T^N}{N!\cdot n^{N\cdot(1-\alpha)}} \qquad (\text{by } (20));$$

i.e.,

$$\otimes \leq \sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \frac{S\left(n^{1-\alpha}\cdot(x-k/n)\right)}{I^*\cdot n^\alpha}\cdot\omega_1\left(f^{(N)},\frac{T}{n^{1-\alpha}}\right)\cdot\frac{T^N}{N!\cdot n^{N\cdot(1-\alpha)}}.$$

Finally, using (6) with the function S instead of b we obtain

$$|R^*| \leq \omega_1\left(f^{(N)},\frac{T}{n^{1-\alpha}}\right)\cdot\frac{T^N}{N!\cdot n^{N\cdot(1-\alpha)}}\cdot\frac{S(T)}{I^*}\cdot\left(2T+\frac{1}{n^\alpha}\right). \quad (35)$$

It is clear now that (34) and (35) imply (33). ∎

Remark 4.

(i) The maps $F_n, G_n, n\in\mathbf{N}$, are positive linear operators.

(ii) Let $x\in[-T,T]$, $S\in C^r([-T,T])$, $r\in\mathbf{N}$, such that $S^{(r)}(x)\geq 0$ over $[-T,T]$. Let $f\geq 0$ and $n\in\mathbf{N}$, $n\geq\max(2T,T^{-1/\alpha})$, $0<\alpha<1$. Then

$$(G_n(f))^{(r)}(x) = \sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{f(k/n)}{I^*}\cdot n^{r-(r+1)\alpha}\cdot S^{(r)}\left(n^{1-\alpha}\cdot\left(x-\frac{k}{n}\right)\right)$$

$$\geq 0,$$

over $[-T,T]$.

REFERENCE

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